

We will learn two ways to multiply vectors.

dot product and cross product.

Name	Sprts out	What it does
Dot product $\vec{u} \cdot \vec{v}$	$\vec{u} \cdot \vec{v}$ is a scalar	Measures how parallel the two vectors are
Cross product $\vec{u} \times \vec{v}$	$\vec{u} \times \vec{v}$ is a vector	Measures how orthogonal the two vectors are

Dot product has a memorable formula.

In 2D, $\vec{u} = \langle a_1, a_2 \rangle$ $\vec{v} = \langle b_1, b_2 \rangle$ then

$$\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2.$$

In 3D, $\vec{u} = \langle a_1, a_2, a_3 \rangle$ $\vec{v} = \langle b_1, b_2, b_3 \rangle$ then

$$\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example

$$\begin{aligned} & \cdot \langle 3, 2, 1 \rangle \cdot \langle -2, 3, 2 \rangle = -6 + 6 + 2 = 2 \\ & \cdot \langle 4, 1, -2 \rangle \cdot \langle 1, 1, 2 \rangle = 4 + 1 - 4 = 1 \end{aligned}$$

In particular, if $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and consider

$\vec{u} \cdot \vec{u}$, it becomes

$$\vec{u} \cdot \vec{u} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= a_1 \cdot a_1 + a_2 \cdot a_2 + a_3 \cdot a_3$$

$$= a_1^2 + a_2^2 + a_3^2 = |\vec{u}|^2.$$

So dot product can be used to obtain the length of a vector.

Some algebra properties (Easy if you write out the formula like above)

$\vec{u}, \vec{v}, \vec{w}$ vectors, a scalar:

- $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

- $\vec{u} \cdot \vec{0} = 0$

- $\vec{u} \cdot \vec{i}$ is the x-component of $\vec{u}$,

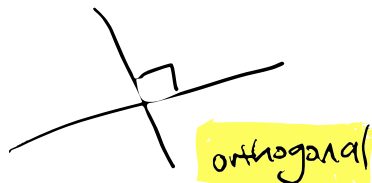
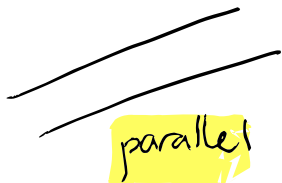
- $\vec{u} \cdot \vec{j}$ is the y-component of $\vec{u}$,

- $\vec{u} \cdot \vec{k}$ is the z-component of $\vec{u}$.

- $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$

Parallelism and orthogonality

We know intuitively what it means for a pair of lines to be **parallel** or **orthogonal** to each other.



In fact, dot product can precisely detect these two notions!

Two vectors $\vec{u}$ and $\vec{v}$ are

- **parallel** to each other if $|\vec{u} \cdot \vec{v}| = |\vec{u}| \cdot |\vec{v}|$.
- **orthogonal** to each other if $\vec{u} \cdot \vec{v} = 0$.

Example

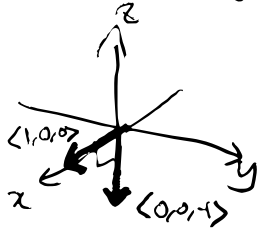
- We know $\vec{u}$ and $-3\vec{u}$ are parallel to each other. Indeed

$$\vec{u} \cdot (-3\vec{u}) = -3(\vec{u} \cdot \vec{u}) = -3|\vec{u}|^2, \text{ so}$$

$$|\vec{u} \cdot (-3\vec{u})| = 3|\vec{u}|^2 = |\vec{u}| \cdot |-3\vec{u}|.$$

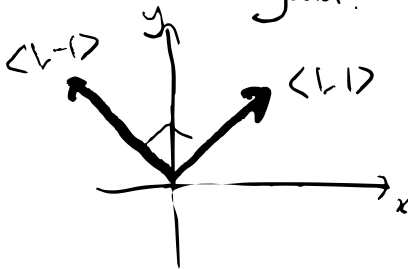
Example, cont'd

- We know $\langle 1, 0, 0 \rangle$ and $\langle 0, 0, -1 \rangle$ should be orthogonal.



$$\text{We check } \langle 1, 0, 0 \rangle \cdot \langle 0, 0, -1 \rangle = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot (-1) = 0.$$

- We know $\langle 1, 1 \rangle$ and $\langle 1, -1 \rangle$ should be orthogonal.



$$\text{We check } \langle 1, 1 \rangle \cdot \langle 1, -1 \rangle = 1 - 1 = 0$$

Two vectors are parallel when the angle between them is either 0° or 180° .

(0 or π in radians)

Two vectors are perpendicular when the angle between them is 90° ($\frac{\pi}{2}$ in radians)

More generally, dot product can measure the angle between two vectors.

Theorem If the angle between two vectors

$\vec{u}$ and $\vec{v}$ is θ (in radians), then

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta.$$

Examples

Indeed, if $\theta = \frac{\pi}{2}$ (or 90°), $\cos \theta = 0$, so

$$\vec{u} \cdot \vec{v} = 0$$

If $\theta = 0$, $\cos \theta = 1$, so

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}|$$

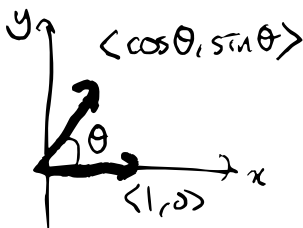
Note that
in both cases
 $\vec{u}$ and $\vec{v}$ are
parallel, since

$$|\vec{u} \cdot \vec{v}| = |\vec{u}| |\vec{v}|$$

If $\theta = \pi$ (or 180°), $\cos \theta = -1$, so

$$\vec{u} \cdot \vec{v} = -|\vec{u}| |\vec{v}|$$

Slightly more complicated: the vector $\langle \cos \theta, \sin \theta \rangle$ form an angle θ with $\langle 1, 0 \rangle$.



We check

$$\langle \cos \theta, \sin \theta \rangle \cdot \langle 1, 0 \rangle$$

$$= \cos \theta$$

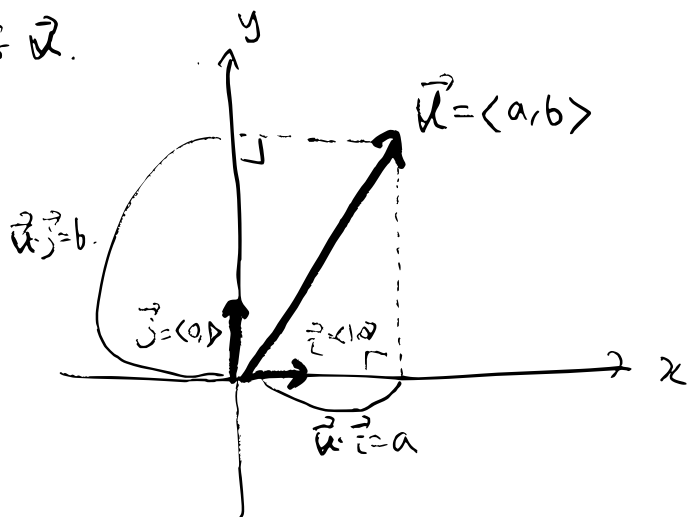
$$|\langle \cos \theta, \sin \theta \rangle| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1$$

$$|\langle 1, 0 \rangle| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1.$$

$$\text{So } \langle \cos \theta, \sin \theta \rangle \cdot \langle 1, 0 \rangle = \cos \theta = |\langle \cos \theta, \sin \theta \rangle| |\langle 1, 0 \rangle| \cos \theta$$

Now we learn how to take the projection of a vector onto another vector.

Remember that $\vec{u} \cdot \vec{i}$ picks the x -component of $\vec{u}$.

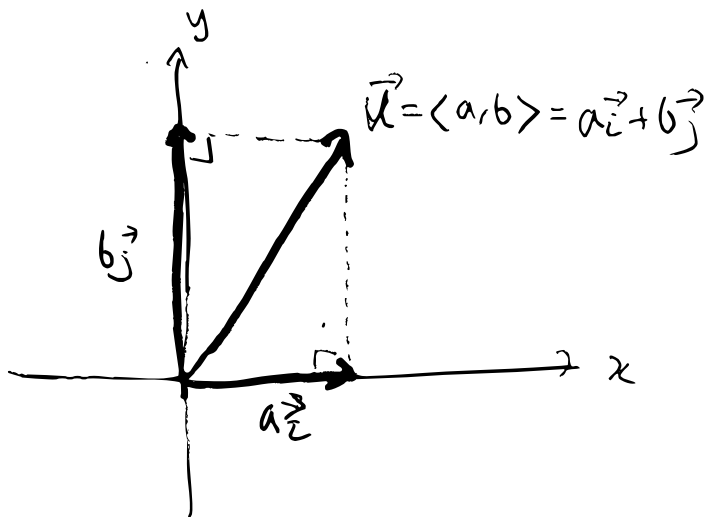


In general, for two vectors $\vec{u}$ and $\vec{v}$,

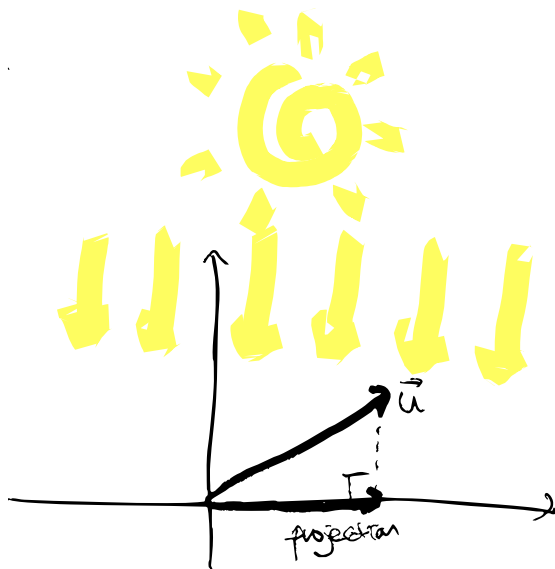
the $\vec{v}$ -direction component of $\vec{u}$ is given by

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}, \text{ denoted as } \text{Comp}_{\vec{v}} \vec{u}.$$

You can furthermore decompose $\vec{u}$ into the sum of a $\vec{i}$ -parallel vector and a $\vec{j}$ -parallel vector, from $\vec{u} = \langle a, b \rangle = a\vec{i} + b\vec{j}$.



The $\vec{i}$ -parallel vector $a\vec{i}$ is called the projection of $\vec{u}$ onto $\vec{i}$. This is like the shadow of $\vec{u}$ on the line of $\vec{i}$.



The projection of $\vec{u}$ onto $\vec{v}$ is denoted $\text{proj}_{\vec{v}} \vec{u}$,
and is given by

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \quad \leftarrow \text{vector}$$

(compare with

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \quad \leftarrow \text{scalar.})$$

Example Let's consider projecting $\vec{u} = \langle 0, 3 \rangle$ onto

$$\vec{v} = \langle 1, 1 \rangle.$$

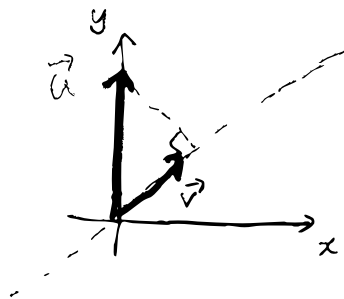
$$\vec{u} \cdot \vec{v} = \langle 0, 3 \rangle \cdot \langle 1, 1 \rangle$$

$$= 3$$

$$|\vec{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{2}}$$

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{3}{(\sqrt{2})^2} \langle 1, 1 \rangle = \frac{3}{2} \langle 1, 1 \rangle = \left\langle \frac{3}{2}, \frac{3}{2} \right\rangle = \langle 1.5, 1.5 \rangle$$



Example for $\vec{u} = \langle 3, -1, 1 \rangle$, $\vec{v} = \langle 4, 7, -4 \rangle$,

$$\vec{u} \cdot \vec{v} = \langle 3, -1, 1 \rangle \cdot \langle 4, 7, -4 \rangle$$

$$= 3 \cdot 4 + (-1) \cdot 7 + 1 \cdot (-4) = 12 - 7 - 4 = 1$$

$$\begin{aligned} |\vec{v}| &= \sqrt{4^2 + 7^2 + (-4)^2} = \sqrt{16 + 49 + 16} \\ &= \sqrt{81} = 9 \end{aligned}$$

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{1}{9}$$

$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{9^2} \langle 4, 7, -4 \rangle = \frac{1}{81} \langle 4, 7, -4 \rangle \\ &= \left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle. \end{aligned}$$